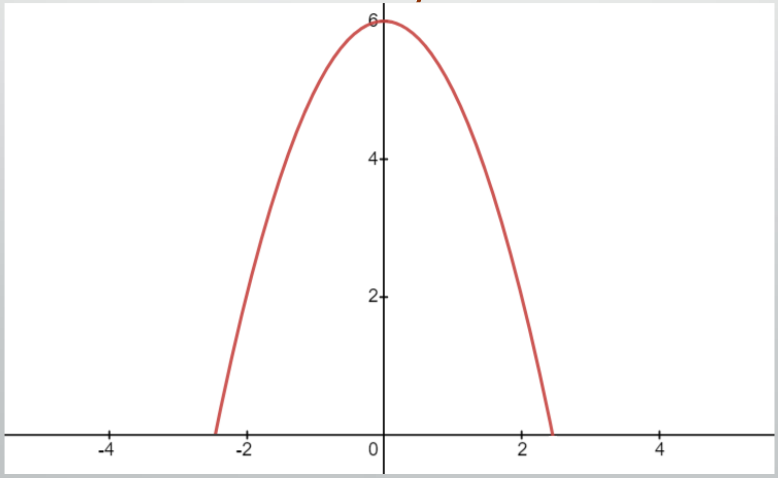
Projectile Motion

Stage 2 physics topic 1.1

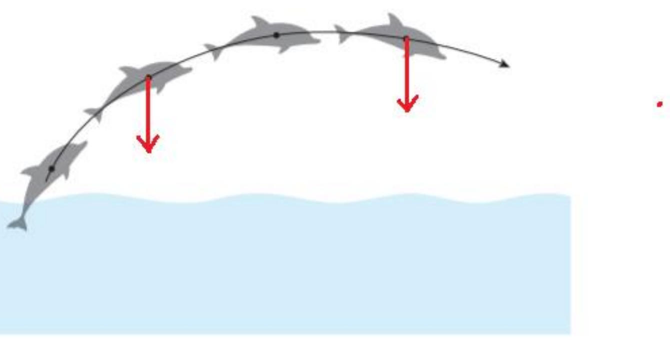
What is a projectile

- A projectile is an object moving through space under the influence of gravity (and maybe friction) **only**
- Unless otherwise informed, assume the only force on it is gravity
  - On Earth, gravity is $9.80ms^{-2}$ down
- Its motion will be a segment of a parabola
  - Note, the 'full' motion will be symmetrical



Only one force - Gravity

- Consider the diagram from the 2019 SACE exam
- The dolphin's position and velocity change throughout its jump
- HOWEVER
- It's acceleration is **always** $9.80 ms^{-2}$ down (as shown by the red arrows)



[This diagram is not drawn to scale.]

The dolphin jumped out of the water with an initial velocity of $9.5\ \mathrm{m\,s^{-1}}$ at an angle of $72°$ above the horizontal.

Ignore air resistance in all parts of this question.

3 Equations of motion

- There are 3 key equations of motion, often attributed to Galileo

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$\vec{s} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}\vec{s}$$

- You MUST learn them

The Equations of Motion

The 3 equations of motion come from the definitions of

average velocity and average acceleration

Average velocity

$$\vec{v} = \frac{\Delta \vec{s}}{\Delta t} = \frac{\vec{v}_0 + \vec{v}}{2}$$

Average acceleration

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v} - \vec{v}_0}{\Delta t}$$

In SACE physics, acceleration is always constant

Equation 1

Equation 1; $\vec{v} = \vec{v}_0 + \vec{a}t$, comes from the definition of acceleration;

Assuming acceleration is constant,

$$\vec{a} = \frac{\vec{v} - \vec{v}_0}{\Delta t}$$

Rearranging

$$\vec{a}\Delta t = \vec{v} - \vec{v}_0$$

Make v the subject

$$\vec{v} = \vec{a}\Delta t + \vec{v}_0$$

Rearrange and it becomes equation 1;

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

Now you can derive equation 1, lets use it

You are driving at $32.0ms^{-1}$ when you see a mountain bike racing ahead of you. Trying desperately to catch the bike you accelerate at $2.50ms^{-2}$ for $5.00s$. What velocity do you get to after the $5.00s$?

Variables

$$v_0 = 32.0ms^{-1}$$

$$\vec{a} = 2.50ms^{-2}$$

$$t = 5.00s$$

Equation

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

Substitution

$$\vec{v} = 32.0 + 2.50 \times 5.00$$

$$\vec{v} = 44.5ms^{-1}$$

Equation 2

We now use average velocity and average acceleration to derive the second equation; $\vec{s} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$

.

Assuming acceleration is constant, our two definition equations are;

$$\vec{v} = \frac{\Delta\vec{s}}{\Delta t} = \frac{\vec{v}_0 + \vec{v}}{2} \quad and \quad \vec{a} = \frac{\vec{v} - \vec{v}_0}{\Delta t}$$

Which we rearrange to become

$$\vec{v} = \frac{2\Delta\vec{s}}{\Delta t} - \vec{v}_0 \ \ldots (1) \ and \quad \vec{v} = \vec{a}\Delta t + \vec{v}_o \ldots (2)$$

Equation 2 cont.

By combining equations (1) and (2) in the previous slide

$$\frac{2\Delta\vec{s}}{\Delta t} - \vec{v}_0 = \vec{a}\Delta t + \vec{v}_0$$

$$\frac{2\Delta\vec{s}}{\Delta t} = \vec{a}\Delta t + 2\vec{v}_0$$

$$2\Delta\vec{s} = \vec{a}\Delta t\Delta t + 2\vec{v}_0\Delta t$$

$$\Delta\vec{s} = \frac{1}{2}\vec{a}\Delta t^2 + \vec{v}_0\Delta t$$

$$\therefore \quad \vec{s} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

Graphical methods for the 2nd equation

Consider a graph of *v* versus t. the displacement covered is the area under the curve (by definition).

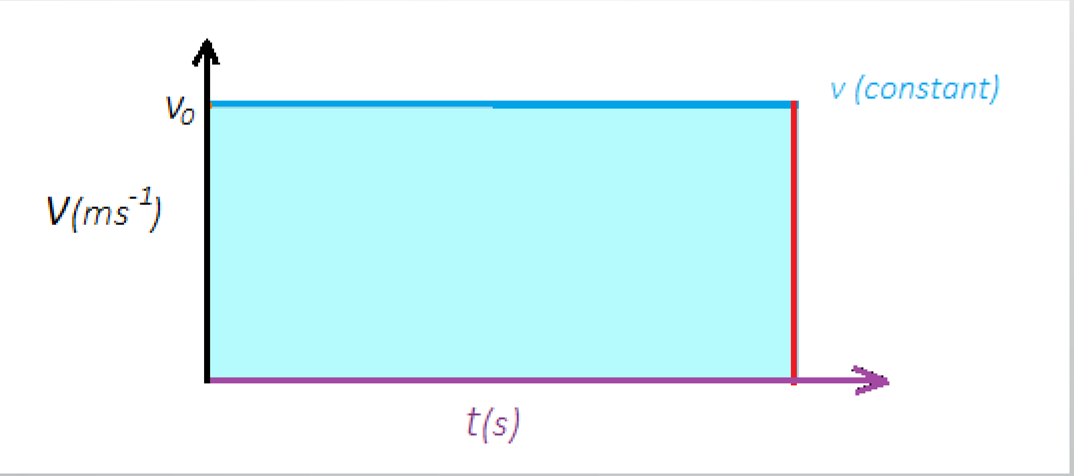
$$s = vt$$

Consider motion at a constant v (the blue line). Let the velocity be v. for every second, the area under the blue line will increase by v (1 second; area = 1v, 2 seconds; area = 2v, etc.)

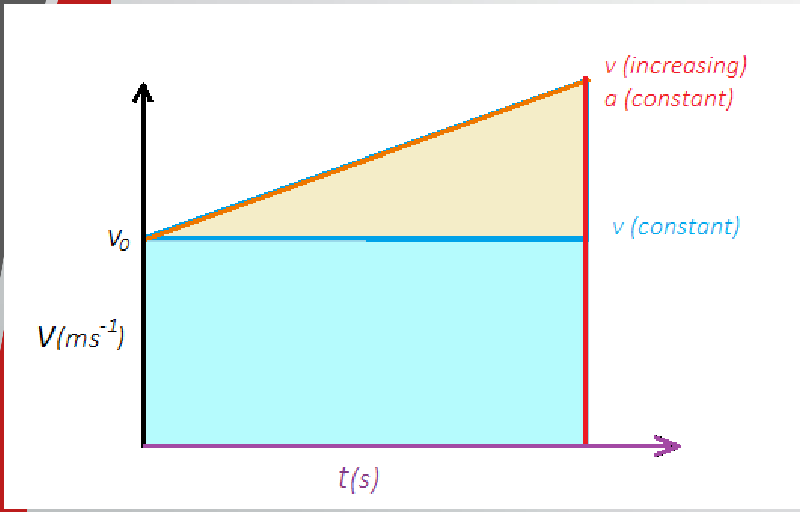
$$\text{since} \quad s = vt$$

Later v will be defined as the initial velocity, so let us call it that now

$$s = v_o t$$



Graphical methods 2



Now consider a motion with constant acceleration as shown by the red line.

Acceleration is defined as

$$\vec{a} = \frac{\Delta\vec{v}}{t}$$

Which is the gradient of the red line

And the area under a triangle (light brown) is

$$area = \frac{1}{2}bh$$

where

$$b = t$$

and

$$h = at\ (since\ a\ is\ the\ gradient)$$

Graphical methods 3

So

$$area = \frac{1}{2} t \, at$$

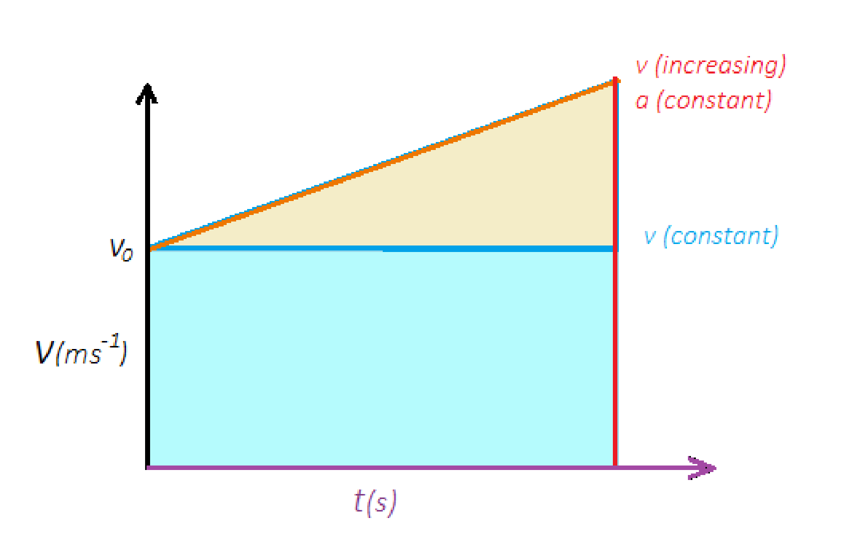
$$area = \frac{1}{2} at^2$$

And since displacement = area under triangle plus area under rectangle

$$\vec{s} = \vec{v}_0 t + \frac{1}{2} \vec{a} t^2$$

Calculus – (optional)

The area under a curve is given by the first integral of the curve



So let the equation of the red curve be of the form

$$y = mx + c$$

Where the y axis is v, the x axis is t

$$m(\text{the gradient}) = a$$

$$c\ (\text{the initial velocity}) = v_o$$

So the equation of the lime (y = mx + c form) is

$$v = at + v_o$$

Which we already know as the first equation of motion

$$v = v_o + at$$

Calculus 2 – (optional)

But onward with calculus, the area under the curve is given by the first integral of the vt curve

$$\int_0^t at + v_o \, dt = \frac{1}{2}at^2 + v_o t$$

Or as we already know it

$$\vec{s} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

Equation 3

$$\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}\vec{s}$$

Going back to our definition equations

$$average \quad \vec{v} = \frac{\Delta\vec{s}}{\Delta t} = \frac{\vec{v}_0 + \vec{v}}{2} \rightarrow \frac{2\Delta\vec{s}}{\vec{v}_0 + \vec{v}} = \Delta t \ldots\ldots\ldots(1)$$

$$\vec{a} = \frac{\vec{v} - \vec{v}_o}{\Delta t} \rightarrow \Delta t = \frac{\vec{v} - \vec{v}_o}{\vec{a}} \ldots\ldots\ldots(2)$$

$$equation\ (1) = equation\ (2)$$

$$\frac{2\Delta\vec{s}}{\vec{v}_o + \vec{v}} = \frac{\vec{v} - \vec{v}_o}{\vec{a}}$$

Equation 3 continued

$$cross\ multiply$$

$$2\Delta\vec{s} \times \vec{a} = (\vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_o) \times (\vec{\boldsymbol{v}}_0 + \vec{\boldsymbol{v}})$$

$$2\vec{a}\vec{s} = \vec{\boldsymbol{v}}^2 - \vec{\boldsymbol{v}}_o^2$$

Rearrange

$$\vec{\boldsymbol{v}}^{\mathbf{2}} = \vec{\boldsymbol{v}}_{\mathbf{0}}^{\mathbf{2}} + \mathbf{2}\vec{\boldsymbol{a}}\vec{s}$$

Now you can derive 3 equations

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$\vec{s} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}\vec{s}$$

From the definitions of average velocity and acceleration

$$\vec{v} = \frac{\Delta\vec{s}}{\Delta t} = \frac{\vec{v}_0 + \vec{v}}{2}$$

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v} - \vec{v}_0}{\Delta t}$$

Gravity

- Gravity acts vertically downwards
  - In fact, "down" is defined by the direction of gravity

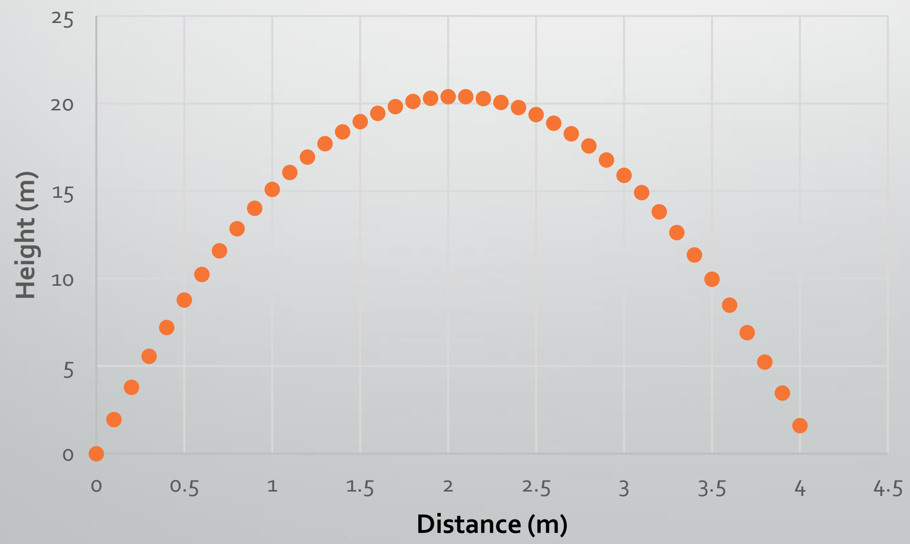
- A mass will accelerate in the direction of gravity in the absence of other forces
  - If air resistance is present, it will act to oppose the forward motion of the mass

- Gravity **$g = 9.80\ ms^{-2}$** vertically down (note 3 sig figs)
  - Note; the "g" variable, "gravity" is given as an acceleration

Brian cox video

Definition

- We call an object that travels a path through air under only the force of gravity a "**projectile**"
- Gravity is the **<u>only</u>** force acting on it during its flight
  - In stage 2 SACE we consider the concept of a projectile also being acted on by the force of air resistance, however, we **DO NOT** perform mathematical calculations in this case.



Missiles are not projectiles

- Despite what many textbook writers and youtubers say
- Missiles use a 'rocket engine'

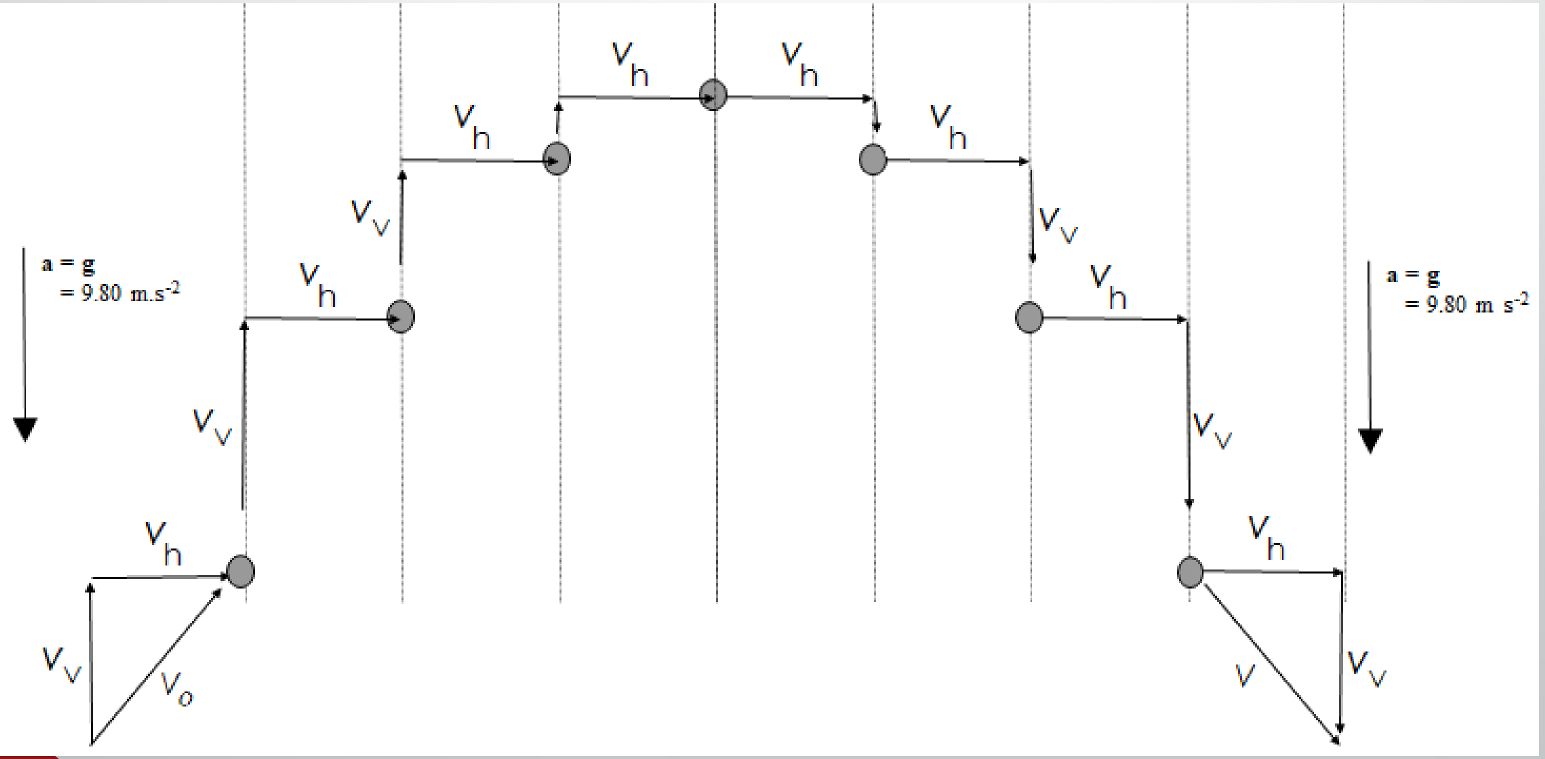
A projectile's path

The following diagram shows the path of a projectile (grey circle) through the air.

Its changing vertical velocity v_v is shown by decreasing then increasing vertical vector arrows

Its constant horizontal velocity v_h is shown by equal horizontal arrows

Acceleration due to gravity is 9.80ms^{-2} DOWN at ALL points in the flight



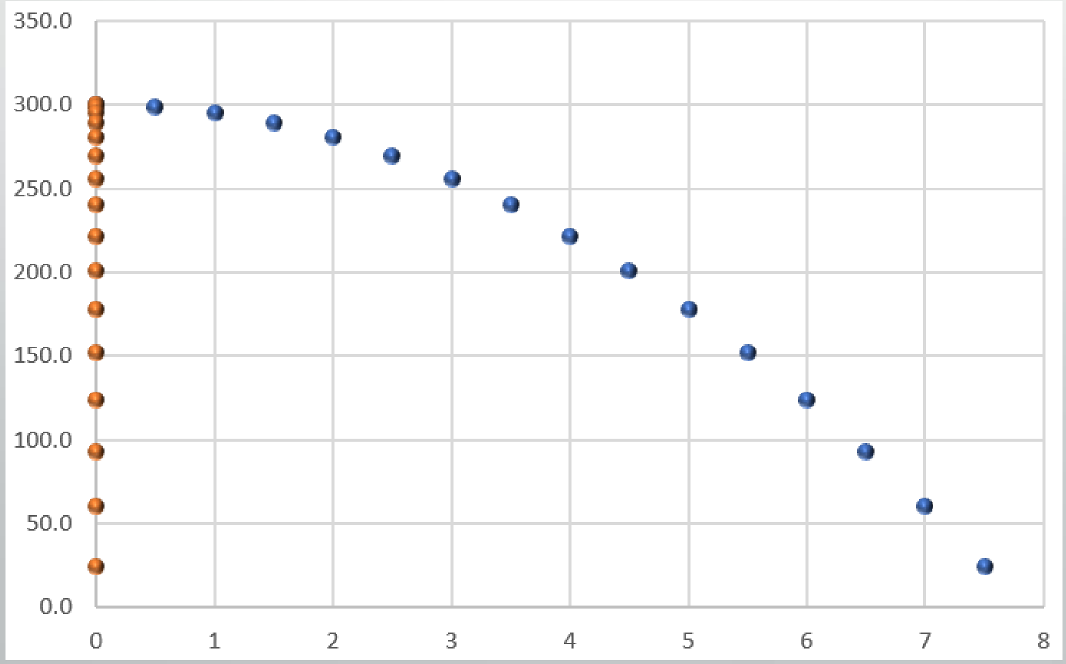
Across and Down

- Horizontal velocity v_h (across) does not change, it is constant.
- Vertical velocity v_v (down) is **ALWAYS** changing at a rate of $9.80ms^{-2}$
- As a result, we ALWAYS consider the two motions separately

Comparing falling objects

Orange object (left) has been dropped, it falls vertically, increasing in velocity. This can be seen as a steady increase in the distance between each position

Blue object (right) started with a horizontal velocity. It continues with that horizontal velocity whilst increasing in its vertical velocity



Projectile motion

Acceleration from gravity is vertical and remains constant at $9.80 ms^{-2}$.

Horizontal velocity remains constant (ignore air resistance)

Vertical motion is independent of horizontal motion.

For now

- The launch height is the same as the impact height
- There is no air resistance

Because vertical motion is independent of horizontal motion, we can determine the time of flight by only considering the vertical motion of the projectile.

Projectile motion

Time of flight (TOF)

Since horizontal and vertical motion are independent, we can use the vertical motion alone to calculate time of flight.

This is where we use our 3 Equations of Motion

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$\vec{s} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}s$$

Simple calculation

- Consider a ball thrown straight up in the air at 10.0ms^{-1}. Since it goes straight up and down, there is no horizontal component.
  - How long till it comes back down (time of flight)?

$$v_0 = 10.0ms^{-1}$$

$a = g = -9.80ms^{-2}$ (minus because it is going down)

We use equation 1

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

At the top of its flight it will stop, just before coming down

So we can set v (final velocity) to zero and calculate ½ the flight

$$\vec{v} = 0$$

Rearrange the equation

$$t = \frac{\vec{v} - \vec{v}_0}{\vec{a}}$$

Simple calculation continued

$$t = \frac{\vec{v} - \vec{v}_0}{\vec{a}}$$

Substitute for variables

$$t = \frac{0 - 10.0}{-9.80}$$

$$t = 1.0204$$

$$TOF\ (time\ of\ flight) = 2 \times t$$

$$TOF = 2.0408$$

$$\underline{TOF = 2.04s}$$

Projectile motion

- Since the acceleration due to gravity is constant (exactly the same on the way up and down)
- the vertical velocities will be equal and opposite (up vs down)
- Thus time to go up = time to come down
- So total time of flight = 2 times the time to get to the highest point.

Did you notice?

- Time to top of flight is

$$\boldsymbol{t = \frac{-\vec{v}_0}{g}}$$

- So … time of flight on a level platform is

$$\boldsymbol{TOF = \frac{-2\vec{v}_0}{g}}$$

Simple calculation - height

- We now know how long it will take to reach the top of its trajectory. We can calculate how high it will go with equation 3

$$\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}s$$

In the vertical direction

$$\vec{v}_0 = 10.0ms^{-1}$$

$$\vec{a} = g = -9.80ms^{-2}$$

$$\vec{v} = 0ms^{-1}$$

Rearrange equation

$$s = \frac{\vec{v}^2 - \vec{v}_0^2}{2\vec{a}}$$

Simple calculation – height, 2

$$s = \frac{\vec{v}^2 - \vec{v}_0^2}{2\vec{a}}$$

$$s = \frac{0^2 - 10.0^2}{2 \times (-9.80)}$$

$$s = \frac{100}{19.6}$$

$$s = 5.1020$$

$$\underline{\underline{s = 5.10m}}$$

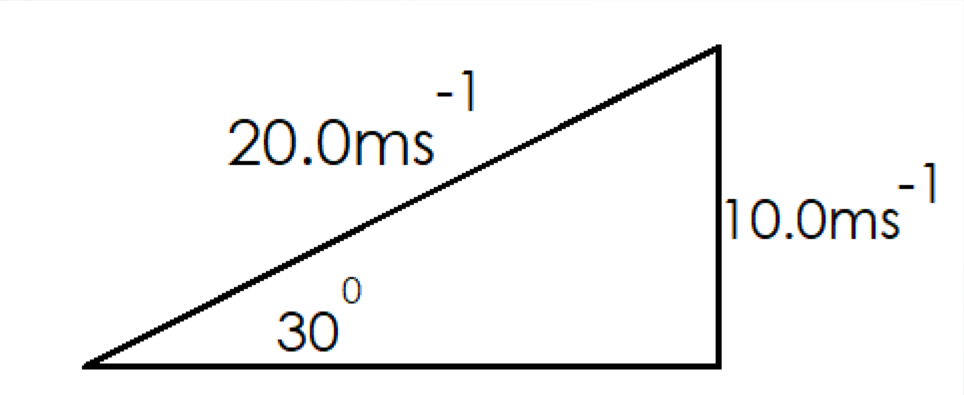
Did you notice

- Maximum height is

$$s = -\frac{\vec{v}_0^2}{2\vec{a}}$$

Slightly harder calculation

- Now consider the ball was thrown at 30° degrees to the horizontal at $20ms^{-1}$



- It still has the same vertical velocity as the first ball
- So, it will still go to the same height and it will still be in the air for the same amount of time

Did you notice?

- It was no harder
- Because the horizontal and vertical components are independent, we treat each one separately
- So the vertical calculations are quite easy
- First step is to calculate the vertical velocity

Consider impact (when it comes down again)

Time of impact occurs when

$$\Delta \boldsymbol{s} = \mathbf{0}$$

Using the second equation of motion

$$\vec{s} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

This has two solutions

The 'trivial solution' $s = 0 \; when \; t = 0$

And also when

$$\vec{v}_0 t = -\frac{1}{2}\vec{a}t^2$$

Impact pt 2

$$\vec{v}_0 t = -\frac{1}{2}\vec{a}t^2$$

$$\vec{v}_0 = -\frac{1}{2}\vec{a}\mathrm{t}$$

$$\frac{2\vec{v}_0}{-\vec{a}} = t$$

$$\mathrm{t} = \frac{2\vec{v}_0}{-g}$$

$(since\ \ \vec{a} = -g\ when\ \mathbf{v}\ up\ is\ positive)$

Range

- 'Range' is the term used to describe the horizontal displacement of a projectile
- Use equation 2

$$\vec{\mathbf{s}} = \vec{\mathbf{v}}_0 \mathbf{t} + \frac{1}{2}\vec{\mathbf{a}}\mathbf{t}^2 \ (\mathbf{note}\ \vec{\mathbf{a}}_{\mathbf{h}} = \mathbf{0})$$

- Thus

$$\vec{s} = v_{h0} t$$

Remembering that the TOF (time of flight) is twice the time to reach maximum height

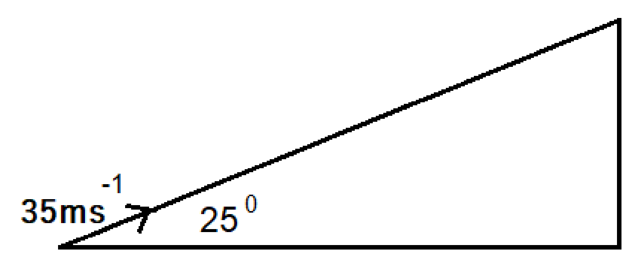
Projectile motion

Example

- Kelly kicks a ball from ground level with a speed of 35 ms^{-1} at an angle of elevation of 25^0 to the horizontal ground surface. Ignoring air resistance determine;
  a. the time the ball is in the air
  b. the horizontal distance travelled by the ball before hitting the ground
  c. the maximum height reached by the ball.

Projectile motion example

Step 1 – variables



$$\vec{v}_0 = 35ms^{-1}$$

$$\theta = 25^0$$

Step 2 – calculate v_v

$$\vec{v}_v = \vec{v}_0 \sin\theta$$

$$\vec{v}_v = 35 \sin 25$$

$$= 14.79ms^{-1}$$ (note; sig figs + 2)

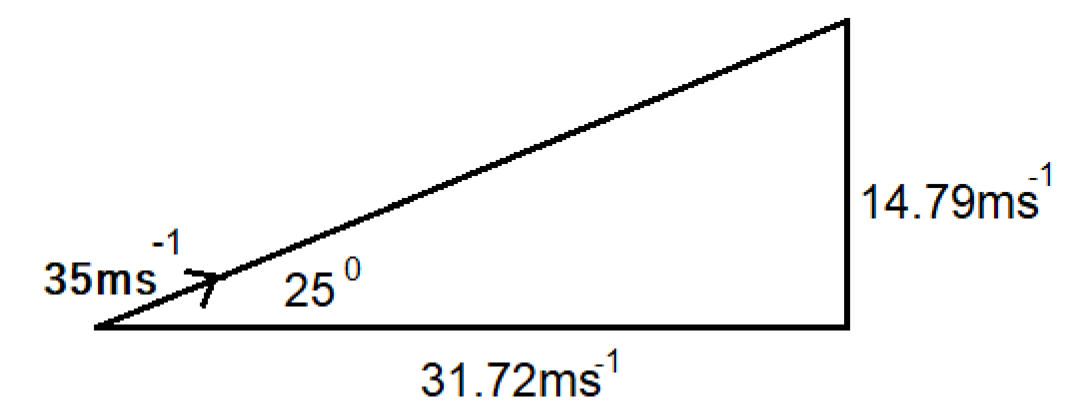
Projectile motion example

Step 3– calculate v_h

$$\vec{v}_h = \vec{v}_0 \cos\theta$$

$$= 35cos25$$

$$= 31.72ms^{-1}$$ (note; sig figs + 2)



Projectile motion example

Using vertical components to determine time to reach maximum height

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_{\mathbf{0}} + \vec{\boldsymbol{a}}\boldsymbol{t}$$

$$where\ \vec{v}_v = 0\ at\ the\ top\ of\ flight$$

$$\therefore t = \frac{\cancel{v_v} - v_{v0}}{a}$$

$$t = \frac{-v_{v0}}{a}$$

Projectile motion example

Time of flight continued

$$\therefore t = \frac{-v_{v0}}{a}$$

$$t = \frac{-14.79}{-9.80}$$

$$t = 1.509s$$

$$\therefore time\ of\ flight = 2t = 3.019$$

$$\underline{\underline{TOF = 3.0s}}$$

Projectile motion

Range

$$s_h = v_{ho}t + \frac{1}{2}a_h t^2 \; where \; a_h = 0$$

$$s_h = v_{ho}t$$

$$s_h = 31.72 \times 3.019$$

$$s_h = 95.76$$

$$\underline{\underline{s_h = 96m}}$$

Projectile motion

Max height (with equation 2)

$$s_v = v_{vo}t + \frac{1}{2}a_v t^2 \; where \; a_v = g$$

$$s_v = 14.79 \times 1.509 + \frac{1}{2} \times 9.80 \times 1.509^2$$

$$s_v = 11.16$$

$$s_v = 11m$$

Projectile motion

Max height (with equation 3)

All variables are vertical components (no subscripts)

$$v^2 = v_0^2 + 2as \ \ where\ a = g, v = 0$$

$$\therefore s = \frac{\cancel{v^2} - v_0^2}{2a}$$

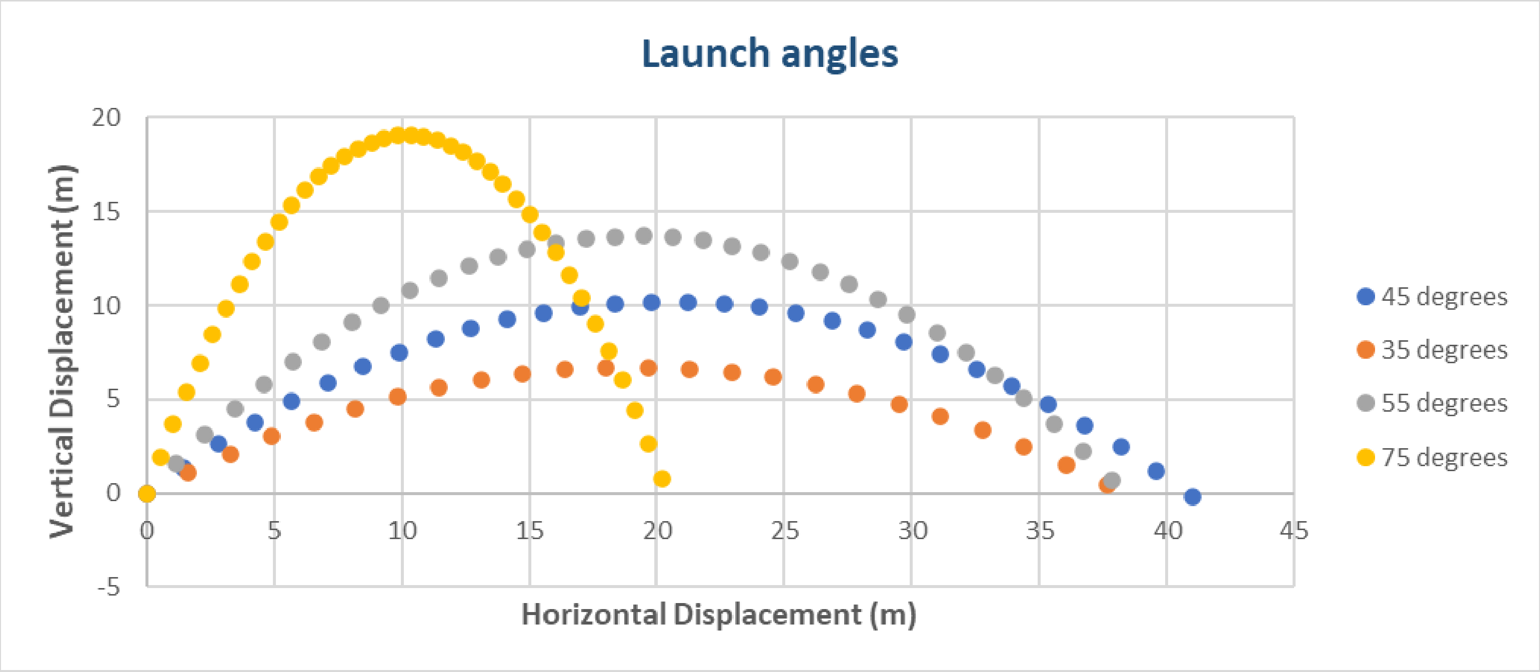
$$s = -\frac{14.79^2}{2 \times 9.80}$$

$$s_v = 11.16$$

$$s_v = 11m$$

Launch angle and range

- Max range is achieved with a 45degree launch angle
- Launch angles that are equal angles either side of 45degrees achieve the same range



BIG NOTE

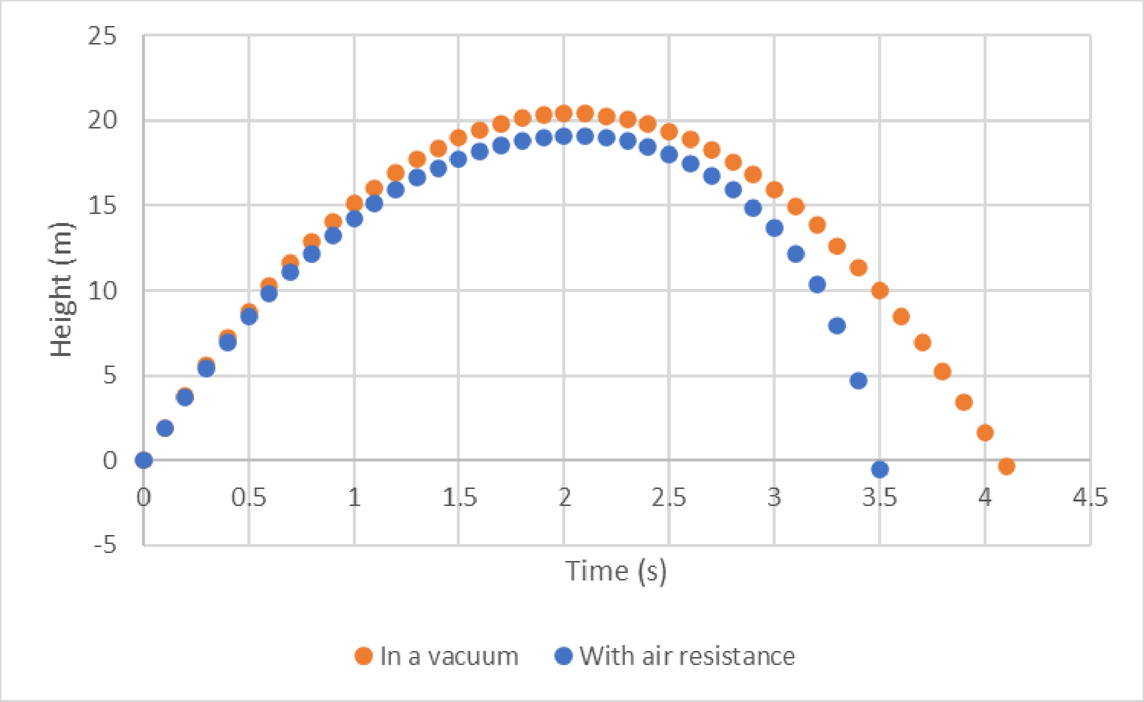
- You do not need to know the mathematics BUT you should know
- Maximum range is achieved when launch angle is 45 degrees from the horizontal
- Equal ranges will be achieved by angles equally either side of 45 degrees.

Air resistance

1. It is a type of friction
2. Affects everything moving through air
3. The force due to air resistance always acts in the opposite direction to the velocity of the object
4. Air resistance is proportional to the speed of the object squared
5. As speed changes, the air resistance must also change

Air resistance

- Friction (air resistance) opposes the direction of motion
- Horizontal velocity is always decreasing
  $(a_h is\ not\ constant)$
- Time of Flight is reduced.
- Range is reduced.



Terminal velocity

- Falling objects will reach a Terminal Velocity
- Terminal velocity is a constant velocity that is the result of friction (air resistance) being equal to the force of gravity
- Therefore Net Force = 0N and by N2L, no force = no acceleration
- It is different for every object and is determined by the shape of the object;
  - Skydivers can change their terminal velocity by changing their shape

Escape velocity

- Whilst we are talking terms, lets mention Escape Velocity
- This is the theoretical velocity required for a projectile to escape Earth's gravitational field
- It is not related to rocket launches in any way
- If a projectile is launched 'straight up' at a velocity of just over 11 000 ms^{-1} it will "reach the apex of its flight" just at the point that gravity goes to zero (0N)